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THE FORMATION OF ROTATIONAL REGIMES IN THE THERMOCAPILLARY FLOW OF A NON-UNIFORM FLUID IN A LAYER†

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The bifurcation of solutions describing thermocapillary flows of a non-uniform fluid in a horizontal layer of finite thickness, under the influence of a temperature gradient at the free boundary is studied. On the assumption that the flow is axially symmetric and has no peripheral component, the velocities of the points of the free boundary are numerically determined as functions of the layer thickness and the temperature gradient. Regions of the parameters are determined in which there are either no solutions, or one or more solutions differing from each other in the form of their velocity profile and the number of flow and counter-flow zones. It is shown for the solutions obtained that at every bifurcation point a pair of new symmetric solutions arises differing from the fundamental solutions by the presence of rotation about the axis of symmetry. Computation of the coefficients of the bifurcation equation reveals the existence of three types of bifurcation point for which the bifurcation equation in the principal approximation contains just two non-zero coefficients. The two-dimensional case of bifurcation is investigated. The bifurcating solutions are constructed asymptotically in the neighbourhood of the bifurcation points and numerically outside such neighbourhoods. © 2000 Elsevier Science Ltd. All rights reserved.

Solutions describing thermocapillary flows due to a temperature gradient along the free boundary have been investigated in numerous publications (see, e.g. [1–6]). Analysis of the bifurcation of unsteady flow modes in a fluid in a thin layer, based on Prandtl's, equations, has shown that bifurcation gives rise to a pair of new modes with rotation about the axis of symmetry [1]. The formation of "self-rotation of fluid" for various classes of flows has been investigated analytically and experimentally [7]. This problem, which has not been studied exhaustively, is related to the formation of waterspouts, tornados, etc. It will be shown below that self-rotation is possible in a fluid with a free layer subject to only radially directed tangential stresses due to non-uniform heating.

1. We consider the problem, formulated for the Oberbeck–Boussinesq equations, of the steady axisymmetric thermocapillary flow of fluid in a horizontal layer of finite thickness bounded below by a solid wall S and above by a free surface Γ, subject to a non-zero longitudinal temperature gradient

$$(\mathbf{v}, \nabla\mathbf{v}) = -\rho^{-1}\nabla p + \nu\Delta\mathbf{v} - \mathbf{g}\beta T \tag{1.1}$$

$$\mathbf{v}\nabla T = \chi\Delta T, \quad \operatorname{div}\mathbf{v} = 0$$

$$p = 2\nu\rho\mathbf{n}\Pi\mathbf{n} - \sigma(k_1 + k_2) + p_*, \quad (r, z, \theta) \in \Gamma$$

$$2\nu\rho[\Pi\mathbf{n} - (\mathbf{n}\Pi\mathbf{n})\mathbf{n}] = \nabla_\Gamma\sigma, \quad T = T_\Gamma, \quad (r, z, \theta) \in \Gamma \tag{1.2}$$

$$\mathbf{vn} = 0, \quad (r, z, \theta) \in \Gamma; \quad \mathbf{v} = 0, \quad T = T_S, \quad (r, z, \theta) \in S$$

Here $\mathbf{v} = (\upsilon_r, \upsilon_\theta, \upsilon_z)$, where r, θ and z are cylindrical coordinates, $\mathbf{g} = (0, 0, -g_t)$, where g_t is the acceleration due to gravity, $\mathbf{n}$ is a unit vector to the free surface Γ, Π is the strain rate tensor, $\nabla_\Gamma = \nabla - (\mathbf{n}, \nabla)\mathbf{n}$ is the gradient along Γ, k_1, and k_2 are the principal curvatures of the free surface Γ, T is the temperature, ν and χ are the coefficient of viscosity and the thermal conductivity, respectively, p_* and T_Γ are the pressure and temperature at the free surface Γ, where $p_* = \text{const}$ and T_s is the wall temperature. The surface tension coefficient σ is assumed to be a linear function of temperature: $\sigma = \sigma_0 - |\sigma_T|(T - T_*)$, where σ_0, σ_T, T_* are known constants and β is the coefficient of thermal expansion. The axial symmetry conditions mean that $\mathbf{v}$, p and T do not depend on the peripheral coordinate θ.

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We will construct a solution of problem (1.1), (1.2) near the axis of symmetry, on the assumption that the region occupied by the fluid is a horizontal layer bounded by known planes: below by a solid wall $z = 0$ and above by a free surface $z = H$. The pressure at the free surface is constant. We will consider the case when the temperature at both boundaries of the layer depends quadratically only on the radial coordinate

$$T_\Gamma = A_\Gamma r^2/2 + c_\Gamma, \quad T_S = A_S r^2/2 + c_S$$

where $A_\Gamma, A_S, c_\Gamma, c_S$ are constants. This corresponds to the special case where there is a layer of stationary thermally conducting air above the free surface, whose temperature satisfies Laplace's equation [6]. Denoting the air temperature at $z = H$ by $T_\Gamma(r)$, we expand the function $T_\Gamma(r)$ in a Taylor series in powers of r. Taking into account that we will henceforth consider the fluid flow only near the axis of symmetry $r = 0$, we will confine ourselves to two terms of the expansion. Noting that $\partial T/\partial r = 0$ for $r = 0$, we obtain the formula $T_\Gamma = c_\Gamma + A_\Gamma r^2/2$ $(r \to 0)$.

We express the solution of system (1.1), (1.2) in the form

$$\begin{aligned}
&v_r = rF'(s)\nu L^{-2}, \quad v_\theta = rG_1(s)\nu L^{-2}, \quad v_z = -2\nu L^{-1}F(s) \\
&T = A_\Gamma(0,5r^2T_1(s) + L^2T_2(s)) \\
&p = -\rho\nu^2L^{-4}(0,5r^2p_1(s) + L^2p_2(s)) \\
&s = z/L, \quad L = (\rho\nu^2|\sigma_T A_\Gamma|^{-1})^{1/3}
\end{aligned} \tag{1.3}$$

where L is a scaling unit of length. We introduce a dimensionless parameter $h = H/L$. The functions (1.3) describe axisymmetric thermocapillary fluid flow in a horizontal layer with solid lower boundary $z = 0$ and free surface $z = H$. Note that this solution describes the flow near the axis of symmetry Oz only and does not extend to the case of large values of the coordinate r.

Let us place the origin of the coordinate system on the free surface, introducing a variable $\xi = 1 - s/h$, and apply a dilatation $F = h^2\Phi(\xi)$, $G_1 = hG(\xi)$. Substituting relations (1.3) into system (1.1), (1.2) and eliminating the pressure, we obtain a non-linear boundary-value problem for the functions Φ, G, T_1

$$\begin{aligned}
&\Phi^{(4)} = 2h^3(\Phi\Phi''' + GG') + \gamma h^2 T \\
&G'' = 2h^3(\Phi G' - \Phi' G) \\
&T_1'' = 2h^3\,\mathrm{Pr}(\Phi T_1' - \Phi' T_1) \\
&\Phi(0) = 0, \quad \Phi''(0) = -1, \quad T_1(0) = 1, \quad G'(0) = 0 \\
&\Phi(1) = \Phi'(1) = G(1) = 0, \quad T_1(1) = \tau
\end{aligned} \tag{1.4}$$

where $\tau = A_S/A_\Gamma$, $\gamma = g\beta A_\Gamma L^5 \nu^{-2}$ are dimensionless parameters, which are assumed to be given; $\mathrm{Pr} = \nu/\chi$ is the Prandtl number.

We will confine our attention to the case $A_\Gamma > 0$ $(\gamma > 0)$ only, corresponding to the case in which the tangential stresses at the free surface are directed toward the axis of symmetry. The functions p_1, p_2 and T_2 are determined after solving boundary-value problem (1.4); p_1, in particular, is found from the formula

$$p_1 = \gamma h \int_0^\xi T_1 d\xi$$

It follows from this relation and from formulae (1.3) that the pressure at the free surface $\xi = 0$ is constant for the solution under consideration.

Let $\Phi_0(\xi, h, \tau, \gamma)$, $\theta_0(\xi, h, \tau, \gamma)$, $G_0 = 0$ denote solutions of system (1.4) that describe fluid flows in which the peripheral component of the velocity vanishes (the fundamental solution). For small h values these solutions are obtained asymptotically by expanding the functions Φ_0 and θ_0 in powers of the parameter h. In particular

$$\Phi_0'(0) = 1/4 + (2\tau + 3)\gamma h^2/240 + O(h^3) \quad (h \to 0)$$

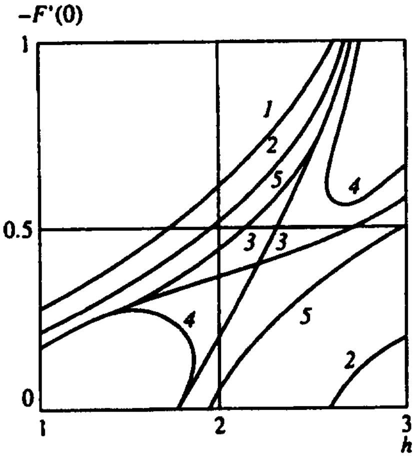

Fig. 1.

For finite values of the parameters h, τ and γ, solutions of system (1.4) have been obtained for γ, = 0, 1; Pr = 7; $h \in [0, 4]$; $\tau \in [-60, 60]$. Figure 1 shows the function $-F'(0)$ as a function of h (note that the quantity $-F'(0)$ is proportional to the radial component of the fluid velocity at the free surface). Curves 1, ... , 5 correspond to the following values of the parameter τ: 50, 0, –41.6072, –50, –30. For each τ value, two branches of the curves have been computed (the layout of the curves in Fig. 1 recalls the picture of the phase trajectories near a saddle-type singular point). Each pair of curves has no common points, with the exception of the case $\tau = -41.6072$.

For each value of $\tau \in [-60, -41.6072]$ there is a range (h_1, h_2) of h values in which there are no solutions. Outside that range two solutions have been computed for each parameter value, but only one solution has been found for $h = h_1$ and $h = h_2$. For example, $h_1 = 1.8195$; $h_2 = 2.4950$ when $\tau = -50$. For $\tau > -41.6072$ two solutions have been computed for each h value. These solutions differ in the shape of the velocity profile. Thus, when $\tau = 0$, for the upper branch 2, the velocity profile has a flow zone near the free surface Γ and a counter-flow zone near the solid wall S. For the lower branch 2, at $h < h_0 - 2.7469$ the velocity profile has two flow zones (one near Γ, the other near S) and a counter-flow zone in between. At $h = h_0$ the fluid has zero velocity at Γ and the flow zone near Γ disappears. At $h > h_0$ the fluid flow zone is near S, while the counter-flow zone is near Γ.

2. We will show that, for certain values of the parameters h and τ, two symmetric solutions with a non-vanishing peripheral component of the velocity $v_\theta \neq 0$ bifurcate from the solution Φ_0, θ_0. To that end, we first consider the eigenvalue problem obtained by linearizing problem (1.4) near the solution Φ_0, θ_0

$$
\begin{aligned}
& f_1^{(4)} = 2h^3(A_0 f_1''' + \Phi_0''' f_1) + \gamma h^2 t_1 \\
& t_1'' = 2h^3 \Pr(\Phi_0 t_1' - \Phi_0' t_1 + f_1\theta_0' - \theta_0 f_1') \\
& g_1'' = 2h^3(\Phi_0 g_1' - \Phi_0' g_1) \\
& \xi = 0: \ f_1 = f_1'' = t_1 = g_1' = 0; \quad \xi = 1: \ f_1 = f_1' = t_1 = g_1 = 0
\end{aligned}
\tag{2.1}
$$

This problem has been investigated numerically for $h \in [0, 4]$. Let h_0 denote the eigenvalues of the parameter h. Obviously, h_0 is a function of the problem parameters τ, Pr and γ. Figure 2 illustrates two branches of the graph of the function $h_0(\tau)$ for Pr = 7 and $\gamma = 0.1$. For every value $-52.3059 \leqslant \tau \leqslant 60$, two eigenvalues h_0 have been computed. For $\tau = -52.3059$ there is a double eigenvalue $h_0 = 2.6089$, but for all other τ values the eigenvalues are simple. The eigenvalues of problem (2.1) have been determined in the form $g_1 = \varphi(\xi), f_1 = 0, t_1 = 0$, with $\varphi(0) = 1$.

We will now derive the bifurcation equation for boundary-value problem (1.4), using the methods of [8, 9], after expressing the solution in the form

$$
\begin{aligned}
& \Phi = \Phi_0 + \alpha f(\xi, h, \tau, \alpha) \\
& G = \alpha g(\xi, h, \tau, \alpha), \quad \theta = \theta_0 + \alpha t(\xi, h, \tau, \alpha)
\end{aligned}
\tag{2.2}
$$

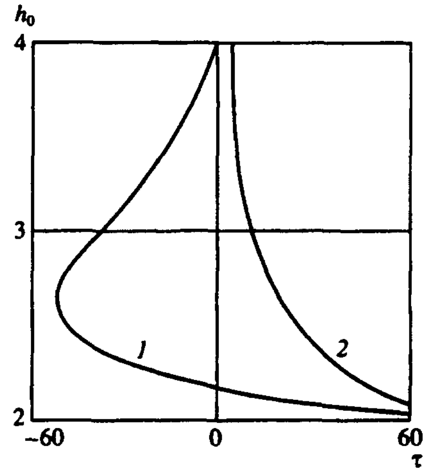

Fig. 2.

where f, g and t are new unknown functions and α is a parameter so chosen that $g = 1$ when $\xi = 0$. We introduce linear operators

$$L = D^4 - 2h^3(\Phi_0 D^3 + \Phi_0'' I)$$
$$K = D^2 - 2h^3(\Phi_0 D - \Phi_0' I)$$
$$N = D^2 - 2h^3 \Pr(\Phi_0 D - \Phi_0' I)$$

where $D = d/d\xi$ and I is the identity operator.

Define a vector $\mathbf{w} = (f, g, t)$. The functions f, g and t are determined by solving the non-linear boundary-value problem

$$\begin{aligned} &L_1\mathbf{w} \equiv Lf - \gamma h^2 t - 2\alpha h^3(ff''' + gg') = 0 \\ &K_1\mathbf{w} \equiv Kg - 2\alpha h^3(fg' - f'g) = 0 \\ &N_1\mathbf{w} \equiv Nt - 2h^3 \Pr(f\theta_0' - \theta_0 f' - 2\alpha h^3 \Pr(ft' - tf') = 0 \\ &\xi = 0: f = f'' = g' = t = 0; \quad \xi = 1: f = f' = g = t = 0 \end{aligned} \tag{2.3}$$

For $\alpha = 0$, $h = h_0(\tau)$, problem (2.3) has the solution $g = \varphi(\xi), f = 0, t = 0$, since it is then identical with eigenvalue problem (2.1).

Consider the Cauchy problem

$$\begin{aligned} &L_1\mathbf{w} = 0, \quad K_1\mathbf{w} = 0, \quad N_1\mathbf{w} = 0 \\ &\xi = 0: \quad f = 0, \quad f' = p_1, \quad f'' = 0, \quad f''' = p_2, \quad t = 0, \quad t' = p_3, \quad g = 1, \quad g' = 0 \end{aligned} \tag{2.4}$$

The parameters p_1, p_2 and p_3 are as yet unknown and will be found by satisfying the boundary conditions in (2.3) at the solid wall, $\xi = 1$.

If $h = h_0$, $\alpha = 0$, the Cauchy problem (2.4) has the solution $g = \varphi(\xi), f = 0, t = 0, p_1 = p_2 = p_3 = 0$. Let us investigate the solution of this problem for parameter values (h, α) near $(h_0, 0)$. These solutions obviously solve boundary-value problem (2.3) if and only if the functions f, g and t satisfy the boundary conditions in (2.3) at $\xi = 1$

$$\begin{aligned} &f(1, \alpha, h, p_1, p_2, p_3) = 0, \quad f'(1, \alpha, h, p_1, p_2, p_3) = 0 \\ &t(1, \alpha, h, p_1, p_2, p_3) = 0, \quad g(1, \alpha, h, p_1, p_2, p_3) = 0 \end{aligned} \tag{2.5}$$

The parameters p_1, p_2 and p_3 are uniquely determined from the first three equations of system (2.5)only if the Jacobian $D(f, f', t)/D(p_1, p_2, p_3)$ does not vanish. Numerical computations have shown that this is the case for all values of h in the range [0, 4] except the point $h = h_* = 2.4806$ and $\tau = \tau_* = -49.2231$, at which the Jacobian vanishes.

Suppose $h \neq h_*$, $\tau \neq \tau_*$. By determining the parameters p_1, p_2 and p_3 from the first three equations of system (2.5) and substituting them into the fourth equation, we obtain the bifurcation equation

$$b(\alpha, h) \equiv g(1, \alpha, h, p_1(\alpha, h), p_2(\alpha, h), p_3(\alpha, h)) = 0 \tag{2.6}$$

Using standard techniques (see [8]), we expand the function $b(\alpha, h)$ in a finite Taylor series in the neighbourhood of the point $\alpha = 0, h = h_0$

$$b(\alpha, h) = b(0, h_0) + (h - h_0)b_h + \alpha b_\alpha + \alpha^2 b_{\alpha\alpha}/2 + \ldots = 0 \tag{2.7}$$

where b_h, b_α, $b_{\alpha\alpha}$ denote the derivatives of $b(\alpha, h)$ with respect to h and α, evaluated at the point $\alpha = 0, h = h_0$.

Let us compute the coefficients of series (2.7). We will first show that $b(0, h_0) = 0$. To that end, we let $\alpha \to 0$, $h \to h_0$ in Eq. (2.6), in the Cauchy problem (2.4), and take into account that $f = f' = t = g = 0$ when $\xi = 1$. Investigation of these problems yields $g = \varphi(\xi)$, $p_1 = p_2 = p_3 = 0$ when $\alpha = 0$ and consequently $b(0, h_0) = \varphi(1) = 0$.

Analogous reasoning yields $b_\alpha = 0$ in (2.7).

We now find the coefficient b_h in (2.7), using the relation

$$b_h = \frac{\partial g}{\partial h} + \sum_{k=1}^{3} \frac{\partial g / \partial p_k}{\partial p_k / \partial h} \quad (\alpha = 0, h = h_0)$$

Note that $\partial g/\partial p_k = 0$ ($\alpha = 0$, $h = h_0$). This follows from an investigation of the Cauchy problem obtained by differentiating the equations and boundary conditions in (2.4) with respect to the parameters p_k. Differentiating Eqs (2.4) with respect to h and letting $\alpha \to 0$, $h \to h_0$, we obtain a Cauchy problem for $g_h = \partial g/\partial h$

$$Kg_h = 2h_0^3(g\Phi'_{0h} - g'\Phi_{0h}) + 6h_0^2(\Phi_0 g' - \Phi'_0 g)$$
$$\xi = 0: \quad g_h = 0, \quad g'_h = 0$$

The function Φ_{0h} is determined from the boundary-value problem

$$L\Phi_{0h} = \gamma h_0^2 \theta_{0h} + 6h_0^2 \Phi_0 \Phi_0''' + 2\gamma h_0 \theta_0$$
$$N\theta_{0h} = 2h_0^3 \Pr(\Phi_{0h}\theta'_0 - \Phi'_{0h}\theta_0) + 6h_0^2 \Pr(\Phi_0\theta'_0 - \Phi'_0\theta_0)$$
$$\xi = 0: \quad \Phi_{0h} = \Phi'_{0h} = \theta_{0h} = 0; \quad \xi = 1: \quad \Phi_{0h} = \Phi'_{0h} = \theta_{0h} = 0$$

The problems obtained for g_h, Φ_{0h}, θ_{0h} are solved numerically. The solid curves in Fig. 3 represent the coefficient b_h against the parameter h. Note that $b_h \to \pm\infty$ as $h \to h_* = 2{,}4806$. In addition,

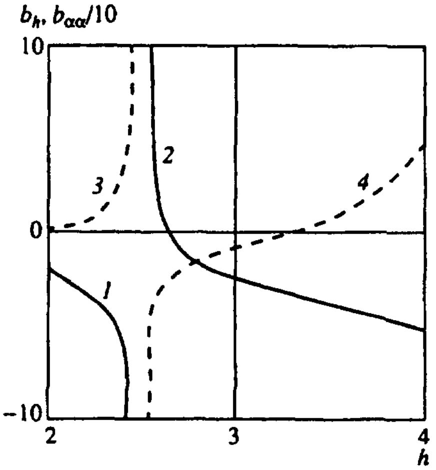

Fig. 3.

$b_h = 0$ when $h = 2.6089$. In the range (2.4806, 2.6089) the coefficient b_h is positive; outside that range it is negative. Curves 1 and 2 in Fig. 3 correspond to τ and h values belonging to the bifurcation branch 1 of Fig. 2. For branch 2 (Fig. 1) the corresponding values of the coefficient b_h are strictly positive (not shown in Fig. 3).

We will now determine the coefficient $b_{\alpha\alpha}$ in (2.7), noting that $b_{\alpha\alpha} = \partial^2 g/\partial\alpha^2$ when $\alpha = 0$, $h = h_0$, $\xi = 1$. The function $g_{\alpha\alpha}$ is found by numerical solution of the Cauchy problem

$$Kg_{\alpha\alpha} = 4h_0^3(f_\alpha g' - f_\alpha' g)$$

$$g_{\alpha\alpha} = 0, \quad g_{\alpha\alpha}' = 0 \quad (\xi = 0)$$

A boundary-value problem for f_α is obtained by replacing the functions f_1, t_1 and g_1 in (2.1) by f_α, t_α, φ, respectively, and adding the term $2h_0^3\,\varphi\varphi'$ to the right-hand side of the first equation. The problem for $g_{\alpha\alpha}$ is solved numerically.

The dashed curves in Fig. 3 (curves 3 and 4) represent graphs of $b_{\alpha\alpha}(h)/10$ as a function of h. Note that $b_{\alpha\alpha} \to \pm\infty$ as $h \to h_* \mp 0$. We have $b_{\alpha\alpha} = 0$ when $h = 3.3419$ and $\tau = -20.831$. Note that $b_{\alpha\alpha} < 0$ in the range (2.4806, 3.3419) and $b_{\alpha\alpha} > 0$ outside that range. Curves 3 and 4 (Fig. 3) were computed for τ and h values belonging to the bifurcation branch 1 of Fig. 2. Plots of $b_{\alpha\alpha}$ for τ and h belonging to the bifurcation branch 2 of Fig. 2 are not shown. Note that the corresponding values of the coefficient $b_{\alpha\alpha}$ are strictly positive.

Now, use of the Newton diagram [9] enables the following parameter value to be obtained from Eq. (2.7)

$$\alpha = \pm\sqrt{2(h_0 - h)b_h / b_{\alpha\alpha}} + \ldots \quad (h \to h_0) \tag{2.8}$$

The coefficient $2b_h/b_{\alpha\alpha}$ in formula (2.8) has been computed numerically.

In Fig. 4 the coefficient $2kb_h/b_{\alpha\alpha}$ is plotted against the parameter h (where k is given the values 0.1, 1 and 10 for branches 1, 2 and 3, respectively). Formula (2.8) is not valid for $h = 3.3419$, where $b_{\alpha\alpha} = 0$. When $h = 2.6089$ we have $b_h = 0$. Consequently, formula (2.8) is again not valid. Note that $b_h \to \pm\infty$, $b_{\alpha\alpha} \to \pm\infty$ as $h \to h_* = 2.4806$, but the quotient $2b_h/b_{\alpha\alpha}$ has a finite limit -0.17365. Thus, formula (2.8) fails to hold in the above three exceptional cases. This analysis will be continued later.

Note that the solutions bifurcate to the side $h < h_0(\tau)$ for values of (τ, h) belonging to the bifurcation branch 2 of Fig. 2, and also for values of (τ, h) belonging to branch 1, in the range $h \in (2.6089, 3.3419)$. Solutions bifurcate toward $h > h_0$ for values of (τ, h) belonging to branch 1 of Fig. 2 in the ranges $h \in (0, 2.6089)$ and $h \in (3.3419, 4)$.

3. We will now construct asymptotic formulae, valid in the neighbourhood of values of $h_0(\tau)$, for solutions that bifurcate from the fundamental solution Φ_0, θ_0 towards $h > h_0$. Points with values $h_* = 2.4806$, $h_1 = 2.6089$, $h_2 = 3.3419$ will be excluded from consideration. Introducing a small parameter $\varepsilon_1 = \sqrt{(h - h_0)}$, we express the solution of problem (1.4) as $\varepsilon_1 \to 0$ in series form

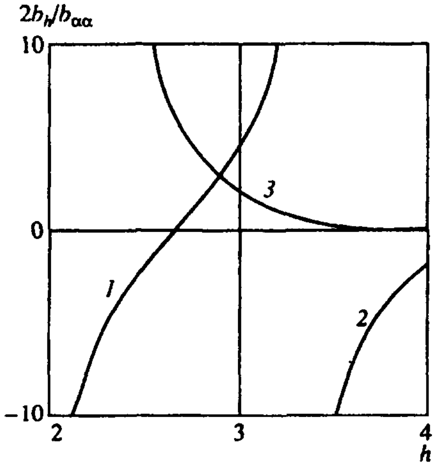

Fig. 4.

$$\Phi = \Phi_{00} + \varepsilon_1 f_1 + \varepsilon_1^2 (f_2 + \Phi_{02}) + \dots$$
$$G = \varepsilon_1 g_1 + \varepsilon_1^2 g_2 + \varepsilon_1^3 g_3 + \dots \quad (3.1)$$
$$T_1 = \theta_{00} + \varepsilon_1 t_1 + \varepsilon_1^2 (t_2 + \theta_2) + \dots$$
$$\Phi_{00} = \Phi_0(\xi, h_0(\tau), \tau, \gamma), \quad \theta_{00}(\xi, h_0(\tau), \gamma)$$
$$\Phi_{02} = \partial\Phi / \partial h, \quad \theta_{02} = \partial\theta / \partial h \quad (h = h_0(\tau))$$

We substitute series (3.1) into system (1.4) and equate the coefficients of ε_1^k ($k = 0, 1, 2, \dots$) to zero. This gives the eigenvalue boundary-value problem for f_1, g_1 and t_1 obtained from (2.1) by replacing Φ_0, θ_0 and h_0 by Φ_{00}, θ_{00}, $h_0(\tau)$, respectively. The solution of this problem is obtained in the form $g_1 = c_1\varphi(\xi)$, $f_1 = 0$, $t_1 = 0$. In the second approximation, the functions f_2, t_2 and g_2 are expressed as $f_2 = c_1^2 f_\alpha$, $t_2 = c_1^2 t_\alpha$, $g_2 = c_2\varphi(\xi)$, where the functions f_α and t_α were determined previously, when computing the coefficient $g_{\alpha\alpha}$ in the bifurcation equation. The coefficients $c_1(\tau)$ and $c_2(\tau)$ are found from the solvability conditions for the boundary-value problems in the third and fourth approximations.

Investigation of the boundary-value problem for the function $g_3(\xi)$ yields the formula $c_1^2 = -2b_h/b_{\alpha\alpha}$, where $b_{\alpha\alpha}$, b_h are the coefficients in formula (2.8), which have already been determined numerically.

Note that the branches below the abscissa axis in Fig. 4 plot the coefficient $-c_1^2$ against the parameter g in this case.

Asymptotic formulae for the solutions that bifurcate from Φ_0, θ_0 for $h < h_0$ are constructed using formulae (3.1) by replacing ε_1 in these formulae by $\varepsilon_2 = \sqrt{(h_0 - h)}$. Now $c_1 = 2b_h/b_{\alpha\alpha}$. The branches above the abscissa axis in Fig. 4 represent c_1^2 against h.

4. At the singular point $\tau = -20.831$, $h = 3.3419$, the coefficients of the bifurcation equation are computed by the standard technique described in [8]; the equation itself may be reduced to the form

$$(h - h_0)b_h + \alpha^4 b_4 / 24 + \dots = 0$$

Numerical computation gives the formula

$$\alpha = \pm 0{,}5136(h - h_0)^{1/4} + \dots, \quad (h \to h_0)$$

Thus, for $\tau = -20.831$, $h_0 = 3.3419$, two symmetric solutions bifurcate from the fundamental solution towards $h > h_0$. The bifurcating solutions are constructed numerically. Computations have shown that when $\tau = -20.831$ the solutions bifurcating from the fundamental solution at $h = 3.3419$ meet again and are identical with the fundamental solution for $h = 2.2590$ at the other bifurcation point.

Curves 1 and 2 in Fig. 5 represent the fundamental solution, and curve 3 a bifurcating solution. The layer thickness h is plotted along the abscissa axis, and $F'(0)$ – which is proportional to the radial component of the velocity at the free surface – along the ordinate axis. A_1 is a singular point at which $h_0 = 3.3419$.

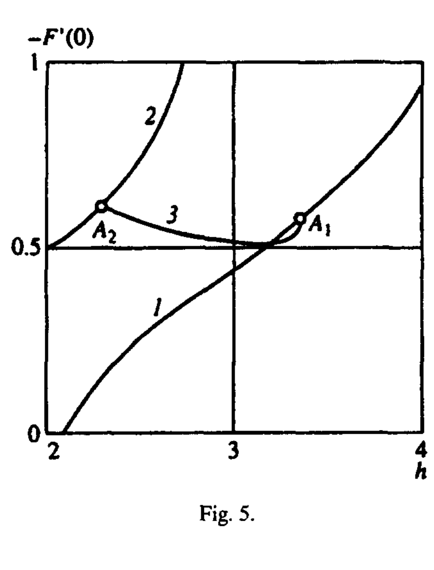

Fig. 5.

5. We will now consider the point with coordinates $h^* = -52.3059$, $\tau^* = -52.3059$, at which $b_h = 0$. Note that here $b_{\alpha h} = 0$. The principal terms of the bifurcation equation are

$$\alpha^2 b_{\alpha\alpha} + (h-h^*)^2 b_{hh} + \ldots = 0 \quad (\alpha \to 0,\ h \to h^*) \tag{5.1}$$

Numerical computation of the coefficients in (5.1) gives $b_{\alpha\alpha} = -30.800$, $b_{hh} = -21.9270$. This means that Eq. (5.1) has no real roots. Consequently, no solutions bifurcate from the point (h^*, τ^*),

Curves 1 and 2 in Fig. 6, plotted in the plane of the parameters $(h, -F'(0))$, respectively represent the fundamental solution and the bifurcating solution for $\tau = -49.22$, which is near τ^*. As $\tau \to \tau^*$ the bifurcation points A_1 and A_2 approach one another, merging at $\tau = \tau^*$, whereupon branch 2 disappears.

6. Let us consider the point for which $h = h_* = 2.4806$ and $\tau = \tau_* = -49.2231$. For the point $D(f, f', t)/D(p_1, p_2, p_3) = 0$, and f, f' and t are determined from (2.5). In addition, $b_h \to \pm\infty$, $b_{\alpha\alpha} \to \pm\infty$ as $h \to h_*$, $\tau \to \tau_*$. Note that at that point the eigenvalue problem (2.1) has a solution different from that described in Section 2. Namely, the eigenfunctions of problem (2.1) are now found in the form $f_1 = \psi(\xi)$, $t_1 = \Gamma(\xi)$, $g_1 = \varphi(\xi)$, with normalization condition $\psi'(0) = 1$. Computations yield the values $\psi'''(0) = -10.8968$ and $\Gamma'(0) = 135.4008$. The function $\psi'(\xi)$ has only one zero in the range $[0, 1]$.

We will now construct asymptotic formulae for the fundamental solution $\Phi_0(\xi)$, $\theta_0(\xi)$ in the neighbourhood of the point (h_*, τ_*). We put $\Phi'(0) = p$ and switch the parameters h and p, that is, consider p as given and h as subject to determination. We define a small parameter $\varepsilon = p - p_*$, where $p_* = 0.2387$ – the value of p at $h = h_*$ and $\tau = \tau_*$. We expand the functions Φ_0, θ_0 and h in powers of ε

$$\Phi_0 = \Phi_{00}(\xi) + \varepsilon\Phi_{01}(\xi) + \ldots, \quad h = h_* + \varepsilon h_1 + \varepsilon^2 h_2 + \ldots \tag{6.1}$$

An analogous series is constructed for the function θ_0, with coefficients $\theta_{00}, \theta_{01}, \ldots$. The functions Φ_{00} and θ_{00} were determined previously from system (1.4) with $h = h_*$ and $\tau = \tau_*$, and the graph of the function $h\Phi'_{00}(0)$ is curve 1 in Fig. 6. Higher-order approximations are found by solving linear boundary-value problems. The principal terms of the asymptotic series as $h \to h_*$, $\tau \to \tau_*$ are

$$\Phi_0(\xi, h) = \Phi_{00}(\xi, h) \pm \sqrt{(h-h_*)/\beta_1}\,\psi(\xi) + O(h-h_*)$$
$$\theta_0(\xi, h) = \theta_{00}(\xi, h) \pm \sqrt{(h-h_*)/\beta_1}\,\Gamma(\xi) + O(h-h_*)$$

The value of $\beta_1 = 83.1127$ was found numerically. Hence it follows that $\partial\Phi_0/\partial h \to \pm\infty$, $\partial\theta_0/\partial h \to \pm\infty$ as $h \to h_*$, $\tau \to \tau_*$.

We will now set up the bifurcation equation for $h = h_*$ and $\tau = \tau_*$. Put $p = \Phi'(0)$ and switch the parameters h and p. The functions Φ, θ and h admit of representations

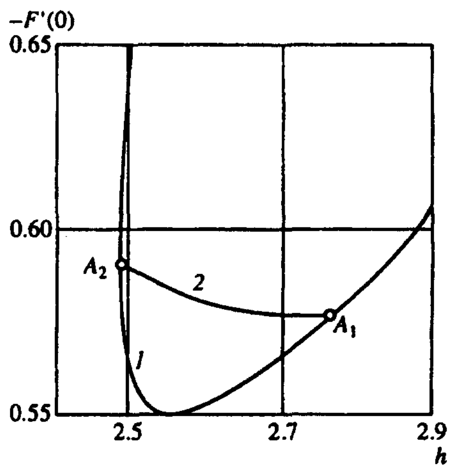

Fig. 6.

$$\Phi = \Phi_0(\xi, p) + \alpha f(\xi, p, \alpha), \quad G = \alpha g(\xi, p, \alpha)$$
$$\theta = \theta_0(\xi, p) + \alpha t(\xi, p, \alpha), \quad h = h_*(p) + \alpha H(\alpha, p) \tag{6.2}$$

given that $g = 0$ ($\xi = 1$), We now derive a non-linear boundary-value problem for the functions f, g and t. Applying the method described in Section 2, we formulate a Cauchy problem to determine the functions f, g and t. Define parameters $p_1 = f'''(0)$, $p_2 = t'(0)$. Now, assuming the boundary conditions to hold at the solid wall ($\xi = 1$), we obtain the relations

$$f(1, \alpha, p_1, p_2, p) = 0, \quad f'(1, \alpha, p_1, p_2, p) = 0$$
$$t(1, \alpha, p_1, p_2, p) = 0, \quad g(1, \alpha, p_1, p_2, p) = 0 \tag{6.3}$$

Numerical computations yield

$$D(f, f', t) / D(p, p_1, p_2) = 0, \quad D(f, f') / D(p_1, p_2) \neq 0$$

We now use the method described in [9] to investigate the two-dimensional case of bifurcation. The bifurcation equation may be reduced to the form

$$(p - p_*) b_p + \alpha^2 b_{\alpha\alpha} / 2 + \ldots = 0$$

The last equation has two solutions

$$\alpha = \pm 1{,}2506 \sqrt{p - p_*} + \ldots \quad (p \to p_*)$$

Thus, two new solutions bifurcate from the point with coordinates $h = h_*$, $\tau = \tau_*$; they differ from the fundamental solution by the presence of a peripheral component of the velocity. Curve 1 in Fig. 6 represents the fundamental solution ($\upsilon_\theta = 0$) and curve 2 represents the bifurcating solution. The point corresponds to the values h_*, τ_*.

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